ELSEVIER

Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of APPLIED MATHEMATICS AND MECHANICS

www.elsevier.com/locate/jappmathmech

Journal of Applied Mathematics and Mechanics 69 (2005) 665–679

EXACT SOLUTIONS OF SOME AERODYNAMIC OPTIMIZATION PROBLEMS†

A. M. YELIZAROV and D. A. FOKIN

Kazan

email: elizgrov@ksu.ru

(*Received* 12 *May* 2004)

Within the limits of the ideal incompressible flow and Chaplygin gas models of the subsonic adiabatic motion of a perfect gas, exact solutions are constructed for the fundamental inverse variational boundary-value problem of aerohydrodynamics, namely, the problem of designing an airfoil of maximum life, on the assumption that the maximum velocity on its contour is limited. The term "variational inverse boundary-value problem" is used to designate a class of two-dimensional boundary-value problems with unknown boundaries, in which it is required to find both the solution of a partial differential equation and its domain of definition, where the latter satisfies some extremal property, and one boundary condition is specified on its boundary. The extremal property of the domain is expressed as the requirement that a certain functional be maximized or minimized (usually with further constraints). The existence and uniqueness of solutions is analysed, admissible domains of the parameters are indicated, examples are given of exact solutions, and an analysis is presented of the tendencies of the aerodynamic shapes being optimized to change when the theoretical angle of attack and maximum value of the velocity on the airfoil contour are varied. The so-called "shelf" distributions of velocity (with sections of constant velocity) are obtained as extremal.

Variational inverse boundary-value problems of aerodynamics constitute one approach to the optimization of aerodynamic shapes. In two dimensions, they consist of designing airfoils that possess optimized characteristics (maximum lift or aerodynamic quality, minimum drag, etc.). These methods make it possible to optimize airfoil shapes and turbomachine cascades in an ideal incompressible fluid, in subsonic gas flow, and in a viscous fluid at high Reynolds numbers. In their formulation, these problems may be classed, on the one had, as optimum design problems (see, e.g. [1]), and on the other as optimization problems for systems with distributed parameters [2]; through the use of the methods of the theory of inverse boundary-value problems they can be reduced to problems of classical variational calculus. At the same time, the presence or absence of further constraints may essentially alter the solvability situation.

When solving variational problems, following the well-known approaches of [3–6], use has been made of the idea of constructing an operator acting on control functions of a given set and control parameters in a given interval, such that for every choice of control functions and parameters there is an object solving the problem and possessing the necessary properties (in the present case, an airfoil bounded by a closed, piecewise-smooth contour).

One corollary of the results obtain by using the theory of inverse boundary-value problems of aerodynamics to solve aerodynamic optimization problems [4–6] is the fact that, among airfoils with one sharp edge and a given length of the contour perimeter, the highest lift in an ideal incompressible flow, uniform at infinity, is that of a disk. Being far from the needs of engineering practice, this solution is obtained analytically, assuming the minimum constraints dictated by the mathematical flow model, and it therefore gives an exact upper limit for the lift. Under physically meaningful conditions (flow without separation taking into account flow viscosity in the boundary-layer approximation, allowance of compressibility of the medium, etc.), even on the assumption that the functionals being minimized are strictly convex, it is impossible to prove that the extremal is unique. The situation becomes even more complicated when the airfoil drag coefficient or aerodynamic quality is the characteristic to be

†*Prikl. Mat. Mekh.* Vol. 69, No. 5, pp. 742–758, 2005.

0021-8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.09.002

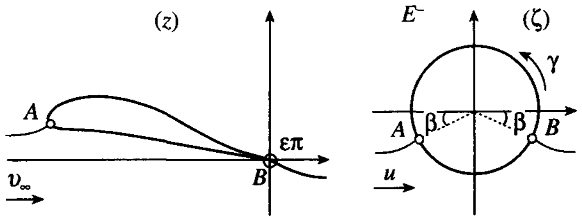

Fig. 1

optimized (even when explicit notation is used). As a result, the optimized solutions are considerably different from a disk, and they can only be computed numerically. However, with certain simplifying assumptions (in particular, a simple choice of the empirical constants in the criteria for flow without separation), one can again obtain strictly convex functionals and construct their extremals (see [6–11]).

Previously investigated problems [6–11] may be categorized as variational inverse boundary-value problems of aerodynamics, and the solutions constructed make allowance, to some degree or another, for the conditions of hydrodynamic feasibility – the conditions of physical feasibility of the solution (the suitability of the mathematical flow model, the assumption that the flow domain is single-sheeted, there is no boundary-layer separation, limitation of the maximum velocity on the contour, etc.) and the solvability conditions (constructive reliability). When that was done, no exact solutions of these problems other than a disk were observed.

One of the natural conditions of hydrodynamic feasibility is to limit the maximum velocity on the contour. Situations have been described in which, when such a limitation is assumed, a unique solution of the variational inverse boundary-value problem exists which is not a disk. Such solutions will be constructed below and an analysis will be made of the tendencies of the optimized aerodynamic shapes to change when the initial, physically meaningful, parameters are varied.

1. FORMULATION OF THE FUNDAMENTAL VARIATIONAL PROBLEM

Among the many possible formulations of variational inverse boundary-value problems of aerodynamics, we shall choose one in which the solution of the problem corresponds directly to one of the common questions of aerodynamics: what is the maximum lift that can be achieved by an airfoil and what is the shape of such an airfoil? We shall present the formulation of the problem for an unbounded ideal incompressible flow.

Physical formulation of the problem. In the plane $z = x + iy$, we consider a steady flow without separation around an impenetrable isolated airfoil whose contour is smooth with the exception of a sharp trailing edge $B(z = 0)$ (Fig. 1). The external angle at the edge is fixed and equal to $\varepsilon\pi$ ($1 \le \varepsilon \le 2$; at $\varepsilon = 1$ the contour is smooth everywhere) and the perimeter of the airfoil contour is $l = 2$. The flow at infinity is uniform, horizontally directed, its velocity is $\upsilon_\infty = 1$ and is density $\rho = 1$. The rear stagnation point of the flow is $z = 0$ (where $\varepsilon > 1$, by the Zhukovskii–Chaplygin hypothesis, this will be sharp edge B). The length scale is taken to be half the contour perimeter (in real airfoils, this is slightly different from the chord length). It is required to determine the airfoil shape yielding the maximum lift coefficient C_y, on the assumption that the maximum velocity on the contour does not exceed a given quantity $\upsilon_{\max}(\upsilon_{\max} > 1)$.

The mathematical model and class of contours to be optimized. We shall now write down the fundamental relations defining the mathematical model of the problem just formulated and the class L of contours to be optimized, following the approach used in [6, 9, 12, 13].

The canonical domain will be the exterior of the unit disk

$$E^- = \{\zeta\colon |\zeta| > 1\}$$

is the auxiliary ζ plane (Fig. 1). We will consider the unit disk in a flow whose velocity vector at infinity, of magnitude u, is directed along the abscissa axis, and such that the critical points $B = e^{-i\beta}$ and $A = -e^{-i\beta}$ on the unit circle (at which the velocity vanishes) are symmetrical about the vertical axis. Here $\beta \in [0, \pi/2]$ is the so-called theoretical angle of attack, which is generally a parameter of the optimization. The value of the angle β may be specified in advance, which imposes an additional constraint on the coefficient C_y in the optimization.

The flow around this airfoil in the physical plane is uniquely defined by the pair consisting of a 2π-periodic control function $P(\gamma) \in L_2[0, 2\pi]$ (where $L_2[0, 2\pi]$ is the space of functions that are square-integrable over the interval $[0, 2\pi]$) that satisfies certain additional smoothness conditions to be specified later, and the parameter β, $\beta \in [0, \pi/2]$. The flow domain is the image of E^- under the conformal mapping $z_p(\zeta)$ normalized so that $z_p(\infty) = \infty$, $z_p(e^{-i\beta}) = 0$.

The coordinates of the desired contour and the optimized functional (the coefficient C_y) are expressed analytically in terms of $P(\gamma)$ and β as follows:

$$x(\gamma) + iy(\gamma) = z_P(e^{i\gamma}) \equiv \frac{2}{J_0(P)} \int_{-\beta}^{\gamma} \exp[P(\tau) + iQ(\tau)] \left|2\sin\frac{\tau+\beta}{2}\right|^{\varepsilon-1} d\tau \tag{1.1}$$

$$Q(\tau) = Q_1(\tau) + (\varepsilon - 1)\frac{\tau+\beta+\pi}{2}, \quad Q_1(\gamma) = -\frac{1}{2\pi}\int_0^{2\pi} P(\tau)\operatorname{ctg}\frac{\tau-\gamma}{2} d\tau$$
$$C_y = 16\pi\sin\beta/J_0(P) \tag{1.2}$$

$$J_0(P) = \int_0^{2\pi} \exp[-P(\tau)] \left|2\sin\frac{\tau+\beta}{2}\right|^{\varepsilon-1} d\tau \tag{1.3}$$

To ensure the existence of the singular integral $Q_1(\tau)$, we require the function $P(\gamma)$ to satisfy a Hölder condition with fixed coefficient and exponent; the set of all such functions is a compact subset of $L_2(0, 2\pi]$. We also note that the derivation of the functional (1.3) makes essential use of the isoperimetric condition $l = 2$ (specification of the perimeter of the unknown contour) (see [5]).

Furthermore, by the choice of $\upsilon_\infty = 1$ and the requirement that the contours be closed, we have the equalities

$$A_0(P) \equiv \int_0^{2\pi} P(\tau)d\tau = B_0, \quad A_1(P) + iA_2(P) \equiv \int_0^{2\pi} P(\tau)\exp(i\tau)d\tau = B_1 + iB_2$$
$$B_0 = 0, \quad B_1 + iB_2 = -\pi(\varepsilon - 1)\exp(-i\beta) \tag{1.4}$$

(note that in the case that the second complex equality (1.4) fails to hold, the openness of the airfoil contour will be greater, the greater the difference between the above integral and the reduced value $B_1 + iB_2$).

The distribution of the magnitude of the velocity over the airfoil contour may be expressed in a parametric form as follows:

$$\upsilon(\gamma) = 2\exp[P(\gamma)]\cos\frac{\gamma-\beta}{2}\left|2\sin\frac{\gamma+\beta}{2}\right|^{2-\varepsilon}$$

For convenience, we assume this quantity to be positive in the interval $\gamma \in [-\beta, \pi + \beta]$ corresponding to the upper surface of the contour, and negative on the lower surface $\gamma \in [\pi + \beta, 2\pi - \beta]$ (where the direction of the flow around the airfoil is the reverse of that of the velocity vector). The requirement that the maximum velocity on the contour be bounded by the given number υ_{max} may now also be expressed in terms of the function $P(\gamma)$ and the parameter β:

$$P(\gamma) \le H_0(\gamma, \beta) \equiv H(\gamma, \beta) + (\varepsilon - 1)\ln\left|2\sin\frac{\gamma+\beta}{2}\right|$$
$$H(\gamma, \beta) = \ln[\upsilon_{max}/M(\gamma, \beta)], \quad M(\gamma, \beta) = |2(\sin\gamma + \sin\beta)| \tag{1.5}$$

The results of many numerical experiments have shown that in the neighbourhood of the exact solution there are various approximate solutions (airfoils with both smooth and sharp trailing edges) that assign the functional to be minimized a value very close to the extremal value, but they differ considerably in geometry from the optimum contours. All this implies that the constructing of an exact solution of the problem is an urgent task.

2. THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

We will express the control function in the form

$$P(\gamma) = T(\gamma) + (\varepsilon - 1)\ln\left|2\sin\frac{\gamma+\beta}{2}\right| \tag{2.1}$$

After substitution into formulae (1.3) and (1.4), we obtain

$$J_0(P) = I(T), \quad I(T) = \int_0^{2\pi} \exp[-T(\tau)]d\tau \tag{2.2}$$

$$A_0(T) = A_1(T) = A_2(T) = 0 \tag{2.3}$$

and the constraint (1.5) becomes

$$T(\gamma) \le H(\gamma, \beta) \tag{2.4}$$

In the functional $I(T)$ and conditions (2.3) the parameter β is not related to the function T. Therefore, when there is no constraint (2.4), the optimum choice would be $\beta = \pi/2$, corresponding to flow around the contour in which the branch point and rear stagnation point coincide. This conclusion is in complete agreement with the well-known fact (see, e.g. [14]) that the maximum velocity circulation in flow around a disk with critical points on its circumference is attained when the forward and rear stagnation points coincide.

Thus, we have arrived at the following variational problem: it is required to determine a 2π-periodic Hölder function $P(\gamma)$ that satisfies condition (1.4), (1.5) and minimizes the functional (1.3). Given the value $\beta = \beta^* > 0$ of the theoretical angle of attack, the problem is equivalent, by conditions (1.4) and (1.5), to the following: for fixed β^* and υ_{max}, it is required to minimize the functional $I(T)$ in the space $L_2[0, 2\pi]$ under conditions (2.3) and (2.4).

It follows from earlier results [5, 6] that $I(T)$ is a strictly convex functional in the space $L_2(0, 2\pi)$, $\inf_{T(\gamma) \in L_2} I(T) = 2\pi$, and this infimum is attained for a unique function $T(\gamma) = T_*(\gamma) \equiv 0$ which does not depend on ε. If $\upsilon_{max} \ge 4$, the function $T_*(\gamma)$ automatically satisfies condition (2.4). In that case the required optimum contour is determined by the mapping $z^*(\zeta) = (\zeta + i)/\pi$ and is a circle of radius $1/\pi$, the flow around which is such that the branch point and rear stagnation point coincide. The absolute maximum C_y^* of the coefficient C_y in Eq. (1.2) is $C_y^* = 8$. Thus, as in the classical isoperimetric problems, in the variational inverse boundary-value problem of aerodynamics under consideration, when $\upsilon_{max} \ge 4$, the extremal is a disk. In the case when $\upsilon_{max} \le 4$ the constraint (2.4) plays an essential role in optimization.

Theorem 1. Let

$$\upsilon^*_{max} = \exp\sin\beta^*, \quad \upsilon^{**}_{max} = 2(1 + \sin\beta^*)$$

A necessary condition for the problem to be solvable is

$$\upsilon_{max} \ge \upsilon^*_{max} \tag{2.5}$$

Moreover, if $\upsilon_{max} \ge \upsilon^{**}_{max}$, the unique extremal is a circle, but if $\upsilon^*_{max} \le \upsilon_{max} < \upsilon^{**}_{max}$, the extremal is not a circle.

If the set U of Hölder functions $P(\gamma)$ satisfying conditions (1.4) and (1.5) is not empty and condition (2.5) is satisfied, the problem has a unique solution.

The assertions of Theorem 1 follow directly from the results of [12, 13], the strict convexity of the functional (1.3), the compactness in the space $L_2[0, 2\pi]$ of the set U (provided it is not empty), and the linearity of conditions (1.4) and (1.5).

By Theorem 1, the problem will have a unique "non-circular" extremal only in the case when $\upsilon^*_{max} \le \upsilon_{max} < \upsilon^{**}_{max}$ and, if υ_{max} is fixed, for $\beta \ge \beta_{max} = \arcsin\ln\upsilon_{max}$. Thus, an admissible domain exists in which the parameters of the problem corresponding to "non-circular" extremals may vary, this domain being bounded above and below by curves whose equations are $\upsilon_{max} = 2(1 + \sin\beta)$ and $\upsilon_{max} = \exp\sin\beta$, respectively. If the point with coordinates (β, υ_{max}) is in that domain, a unique optimum airfoil exists that is not a disk. If the point lies above that domain, the solution of the variational problem will be a disk, and if it is below the domain, there is no solution.

3. CONSTRUCTION OF AN EXACT SOLUTION

The form of the extremal function $P^*(\gamma)$ enables us to establish the Kuhn–Tucker theorem (see, e.g. [15, Section 1.1.2]). Consider the extended functional

$$\Psi(P) = \int_0^{2\pi} F(P,\tau)d\tau \equiv J_0(P) + \mu_0[A_0(P)-B_0] + \mu_1[A_1(P)-B_1] + \mu_2[A_2(P)-B_2] +$$
$$+\int_0^{2\pi} \mu(\tau)[P(\tau)-H_0(\tau,\beta)]d\tau$$

The parameters μ_0, μ_1 and μ_2 must be fixed in such a way that conditions (1.4) are satisfied, and $\mu(\gamma)$ is a non-negative function needed to guarantee the truth of condition (1.5). By the necessary condition for an extremum, the form of the extremal function $P^*(\gamma)$ is determined by the equation $\partial F/\partial P = 0$:

$$\begin{aligned} P^*(\gamma) &= (\varepsilon-1)\ln\left|2\sin\frac{\gamma+\beta}{2}\right| - \ln g(\mu_k,\mu^*;\gamma) \\ g(\mu_k,\mu^*;\gamma) &\equiv \mu_0+\mu_1\cos\gamma+\mu_2\sin\gamma+\mu^*(\gamma) \end{aligned} \tag{3.1}$$

where the parameters $\mu_k (k = 0, 1, 2)$ and the function $\mu^*(\gamma)$ are such that $g(\mu_k, \mu^*; \gamma) \geq 0$. The non-negative function $\mu^*(\gamma)$ is found from the so-called additional non-rigidity condition $\mu^*(\gamma)\ [P^*(\gamma) - H_0(\gamma, \beta)] = 0$

$$\mu^*(\gamma) = \max\{0, v_{\max}^{-1}M(\gamma,\beta)-\mu_0-\mu_1\cos\gamma-\mu_2\sin\gamma\} \tag{3.2}$$

The velocity distribution corresponding to the extremal function $P^*(\gamma)$ is

$$|v^*(\gamma)| = \min\left\{v_{\max};\left|\frac{M(\gamma,\beta)}{\mu_0+\mu_1\cos\gamma+\mu_2\sin\gamma}\right|\right\} \tag{3.3}$$

The minimum of the functional is

$$J^* = J_0(P^*) = 2\pi\mu_0 + \int_0^{2\pi}\mu^*(\tau)d\tau > 0 \tag{3.4}$$

Conditions (1.4) may be rewritten for the extremal function $P^*(\gamma)$ as

$$\int_0^{2\pi}\ln g(\mu_k,\mu^*;\gamma)d\gamma = \int_0^{2\pi}\ln g(\mu_k,\mu^*;\gamma)\exp(i\gamma)d\gamma = 0 \tag{3.5}$$

Note that the quantity J^*, the function $\mu^*(\gamma)$, $v^*(\gamma)$, and Eqs (3.5) do not contain the quantity ε, which determines the opening span of the angle of the airfoil at the trailing edge. Thus, the extremal we have obtained is the same for airfoils with sharp ($\varepsilon > 1$) and blunt ($\varepsilon = 1$) edges. However, as may be seen from Eqs (3.1), in the first case the function $P^*(\gamma)$ has a logarithmic singularity at $\gamma = -\beta$; consequently, the solution does not fall into the given class and gives only an upper limit for the maximum C_y. But in the case of a blunt edge ($\varepsilon = 1$) one can construct an airfoil for which this maximum is attained.

By virtue of relations (1.3), (3.1) and (3.4), the exterior of the unit disk is mapped conformally onto the domain of flow around an airfoil of optimum shape by a function $z_{p*}(\zeta)$ of the form

$$z^*(\zeta) = \frac{2}{J^*}\int_{e^{-i\beta}}^{\zeta}\exp[G(\zeta)]d\zeta \tag{3.6}$$

where $G(\zeta) = (\mathbf{S}\ln g)(\zeta)$ is analytic in E^-, its real branch is such that $\operatorname{Re}G(e^{i\gamma}) = \ln g(\mu_k, \mu^*; \gamma)$ on the circle, and $\operatorname{Im}G(\infty) = 0$; $\mathbf{S}$ is the Schwartz operator. In the special case when $\mu^*(\gamma) \equiv 0$ (in the absence of constraint (1.5)), representation (3.6) implies the previously obtained representation $z^*(\zeta) = (\zeta + i)/\pi$.

We emphasize once more that the parameter ε does not occur informula (3.6), and the images of the domain E^- under the mappings (3.6) for different β and υ_{max} will have smooth boundaries. If $\varepsilon = 1$, that will be the domain of flow about an airfoil of optimum shape. It will be shown below that for any β and υ_{max} these domains are symmetric about the vertical but not always single-sheeted.

4. THE SYMMETRY OF THE OPTIMUM SOLUTION

Theorem 2. If the necessary condition for solvability (2.5) is satisfied, then $\mu_1 = 0$, $\mu^*(\gamma) = \mu^*(\pi - \gamma)$, the velocity distribution (3.3) increases monotonically over the interval $\gamma \in [-\pi/2, \pi/2]$, and if $\varepsilon = 1$ the optimum contour has a vertical axis of symmetry.

Proof. Let us assume that we have successfully determined a set of parameters μ_0, $\mu_1 > 0$, μ_2 satisfying the solvability conditions (3.5). We will show that in that case the set of parameter μ_0, $-\mu_1 < 0$, μ_2 also determines a solution of the problem.

Making the change of variables $\tau = \pi - \gamma$ in Eqs (3.5), we obtain

$$\int_0^{2\pi} \chi(\gamma)d\gamma = 0, \quad \int_{\pi}^{-\pi} \chi(\gamma)\cos\gamma d\gamma = 0, \quad \int_0^{2\pi} \chi(\gamma)\sin\gamma d\gamma = 0$$

where

$$\chi(\gamma) = \ln|\omega(\gamma) - \mu_1\cos\gamma + \tilde{\mu}(\gamma)|, \quad \omega(\gamma) = \mu_0 + \mu_2\sin\gamma$$

$$\tilde{\mu}(\gamma) = \mu^*(\pi - \gamma) = \max\{0, \upsilon_{max}^{-1}M(\gamma, \beta) - \omega(\gamma) + \mu_1\cos\gamma\}$$

To verify that the velocity function $\upsilon_1^*(\gamma)$ corresponding to the new set of parameters satisfies the required boundedness condition, we write the following chain of relations

$$|\upsilon_1^*(\gamma)| = \left|\frac{M(\gamma, \beta)}{\omega(\gamma) - \mu_1\cos\gamma}\right| = \left|\frac{M(\tau, \beta)}{\omega(\tau) + \mu_1\cos\tau}\right| = |\upsilon^*(\tau)| < \upsilon_{max}$$

Thus, the velocity indeed satisfies the boundedness condition. Finally, after substituting the modified set of parameters μ_0, $-\mu_1$, μ_2 and the corresponding function $\bar{\mu}(\gamma)$ into the necessary condition for an extremum, we can convince ourselves that these values also make the functional J_0 a global minimum. Thus, the set μ_0, $-\mu_1 < 0$, μ_2 also determines a solution of the extremal problem under consideration, contrary to the uniqueness of the solution. Therefore $\mu_1 = 0$. Hence it also follows that $\mu^*(\gamma) = \mu^*(\pi - \gamma)$, so that it will suffice to confine our attention to the interval $\gamma \in [-\pi/2, \pi/2]$, with the solution on the rest of the circle determined by symmetry. A corollary of this property is also that the part of the solvability conditions (3.5) containing $\cos\tau$ as a weighting function is also satisfied. If $\varepsilon = 1$, the equality $\mu_1 = 0$ ensures that the corresponding optimum contour will symmetric about the vertical axis, as is readily verified by directly substituting the extremal function $P^*(\gamma)$ into Eq. (3.6).

Further, since $\mu_1 = 0$ and the velocity on the upper surface is positive and that on the lower surface negative, it follows that $\omega(\gamma) \geq 0$ for $\gamma \in [-\beta, \pi + \beta]$ (see (3.3)). Hence

$$\omega(-\beta) = \mu_0 - \mu_2\sin\beta \geq 0$$

Then

$$\upsilon^{*\prime}(\gamma) = \left[\frac{2(\sin\gamma + \sin\beta)}{\omega(\gamma)}\right]' = 2\frac{\mu_0 - \mu_2\sin\beta}{\omega(\gamma)^2}\cos\gamma \geq 0, \quad \gamma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Thus, the velocity distribution increases monotonically.

Corollary. If

$$2(1 + \sin\beta) > \upsilon_{max}(\mu_0 + \mu_2) \tag{4.1}$$

then in the interval $[t, \pi/2]$ where

$$t = \arcsin[(\mu_0 v_{max} - 2\sin\beta)/(2 - \mu_2 v_{max})]$$

the velocity distribution $v^*(\gamma)$ has a "shelf" $v = v_{max}$.

Computational experiments have shown that, in optimum shapes corresponding to the distribution (3.3) with $\beta > 0$, "shelves" cannot arise simultaneously on the upper and lower surfaces, though this still lacks a rigorous proof. If the hypothesis is true, then always

$$v^*(\gamma) = v_1^*(\gamma) \equiv \frac{2(\sin\gamma + \sin\beta)}{\omega(\gamma)}, \quad \gamma \in \left[-\frac{\pi}{2}, t\right]; \quad v^*(\gamma) = v_2^*(\gamma) \equiv v_{max}, \quad \gamma \in \left[t, \frac{\pi}{2}\right] \tag{4.2}$$

with $t = \pi/2$ if the reverse inequality to (4.1) is true, and then system for determining the parameters, taking the symmetry of the optimum velocity distribution into account, becomes

$$\int_{-\pi/2}^{\pi/2} \ln|v^*(\gamma)|d\gamma = 0, \quad \int_{-\pi/2}^{\pi/2} \ln|v^*(\gamma)|\sin\gamma d\gamma = \pi\sin\beta \tag{4.3}$$

Thus, the solution of the problem has been reduced to solving the system of non-linear equations (4.3) for determining the two unknowns μ_0 and μ_2.

If constraint (1.5) is relaxed, then $\mu^*(\gamma) \equiv 0$ and system (4.3) becomes

$$K_0 \equiv \int_0^{2\pi} \ln|\omega(\gamma)|d\gamma = 0, \quad K_1 \equiv \int_0^{2\pi} \ln|\omega(\gamma)|\sin\gamma d\gamma = 0$$

Evaluation of the integrals K_0 and K_1 yields two solutions: $\mu_2 = 2$, $\mu_0 = 0$ and $\mu_2 = 0$, $\mu_2 = 1$. Since the first solution does not ensure that the function $g(\mu_k, \mu^*, \gamma)$ will have a fixed sign, it may be ignored. Finally, we have

$$g(\mu_k, \mu^*; \gamma) \equiv 1, \quad P^*(\gamma) = (\varepsilon - 1)\ln\left|2\sin\frac{\gamma + \beta}{2}\right|, \quad J^* = 2\pi$$

The corresponding function is $T^*(\gamma) \equiv 0$ (see (2.1)), and the optimum contour is the circle defined by the mapping $z^*(\zeta) = (\zeta + i)/\pi$. Thus, we have again obtained an extremal solution in the form of a disk (compare the results of [5]).

5. INVESTIGATION OF THE SOLVABILITY OF SYSTEM (4.3)

First of all, let us consider the case $\mu_2 \neq 0$. Put $m = \mu_0/\mu_2$. Transform the system of equations (4.3) using the following expressions for the integrals K_0 and K_1, obtained by evaluating them as contour integrals for $|m| \leq 1$ and by differentiation with respect to m and then reduction to tabulated integrals for $|m| > 1$

$$K_0 = 2\pi\ln|\mu_2/2| + 2\pi T_0(m), \quad K_1 = 2\pi m - 2\pi T_1(m)$$

where

$$T_0(m) = \mathrm{sign}(m)\ln\left|m + \sqrt{m^2 - 1}\right|, \quad T_1(m) = \mathrm{sign}(m)\sqrt{m^2 - 1}, \quad \text{if } |m| > 1$$
$$T_0(m) = T_1(m) = 0, \quad \text{if } |m| \leq 1$$

On the assumption that the formation of a "shelf" is possible only on the upper surface of the contour of the optimized airfoil, substituting expression (4.2) into Eq. (4.3) we obtain

$$\Phi_0(t, m) = R_0(t, m), \quad \Phi_1(t, m) = R_1(t, m) \tag{5.1}$$

$$\Phi_0(t,m) \equiv \frac{\cos t}{\pi/2+t}F_0(t,m) + F_1(t,m), \quad \frac{1}{\pi}R_0(t,m) \equiv m - T_1(m) + \frac{\cos t}{\pi/2+t}T_0(m)$$

$$\Phi_1(t,m) \equiv \exp\frac{F_0(t,m) - \pi T_0(m)}{\pi/2+t}, \quad R_1(t,m) \equiv \frac{2}{\upsilon_{\max}}\left|\frac{\sin t + \sin\beta}{\sin t + m}\right|$$

where

$$F_0(t,m) = \pi\ln 2 - \int_t^{\pi/2} \ln R_1(\gamma,m)d\gamma, \quad F_1(t,m) = -\int_t^{\pi/2} \ln R_1(\gamma,m)\sin\gamma d\gamma$$

Note that the condition $\upsilon^*(t) = \upsilon_{\max}$ implies an explicit expression for the unknown μ_2 in terms of the parameters t, m:

$$\mu_2 = \frac{2(\sin t + \sin\beta)}{\upsilon_{\max}(\sin t + m)} \tag{5.2}$$

By analysing the limits between which the unknown parameters μ_0 and μ_2 are allowed to vary, the following conditions on m have been established

$$\begin{aligned} &-b_+ - 1 \le m \le 1 - b_- \quad \text{for} \quad \mu_2 < 0 \\ &1 + b_- \le m \le b_+ - 1 \quad \text{for} \quad 0 < \mu_2 \le a\min\{\sin\beta, 1 - \sin\beta\} \\ &b_- - 1 \le m \le 1 - b_- \quad \text{for} \quad a(1-\sin\beta) < \mu_2 \le a \end{aligned} \tag{5.3}$$

where

$$a = 2/\upsilon_{\max}, \quad b_\pm = a(1 \pm \sin\beta)/|\mu_2|; \quad b_- \le b_+$$

Note also that, by Theorem 1, the following inequalities are true for extremals that are not circles

$$(1+\sin\beta)^{-1} \le a \le 2\exp(-\sin\beta)$$

Thus, we have to find a pair (m^*, t^*), a solution of the system of equations (5.1), where the intervals within which m varies in (5.3) depend on μ_2, which is determined by the solution (m^*, t^*) from formula (5.2). Note that

$$[1+b_-, b_+ - 1] \subset (1,\infty), \ [b_- - 1, 1 - b_+] \subset (-\sin\beta, \sin\beta), \ [-1-b_+, 1-b_-] \in (-\infty, \sin\beta)$$

Investigations have shown that for any fixed $t \ge 0$ the function $\Phi_0(t, m)$ increases monotonically as a function of m for $m > -1$, decreases monotically for $m < -1$, and is convex from above in both cases. Moreover, as $|m| \to \infty$,

$$\Phi_0(t,m) = \pi(\pi/2+t)^{-1}\cos t\ln|m| + O(|m|)$$

Thus, at infinity the function $\Phi_0(t, m)$ increases logarithmically. It is obvious from the representation of the function $R_0(t, m)$ (see (5.1)) that the behaviour at $\pm\infty$ of the function $R_0(t, m)$ is exactly the same (with the same coefficient of the logarithm). When $|m| < 1$ the function $R_0(t, m)$ is linear. When $m > 1$ the function $R_0(t, m)$ first decreases monotonically, reaching a minimum at $m = m^* = (\pi/2 + t)/\cos t$ and remaining convex from below, but then increases monotonically and becomes convex from above. The function $\Phi_1(t, m)$ is strictly positive for any t, bounded, and consists of two branches, each of which tends to zero as $m \to \pm\infty$, remaining convex from below. The function $R_1(t, m)$ also consists of two symmetric branches with an asymptote at $m = -\sin t$ (as this value is approached from either side, $R_1(t, m)$ increases without limit). As $m \to \pm\infty$ the function $R_1(t, m)$ tends to zero.

The properties of the functions $\Phi_0(t, m)$ and $R_0(t, m)$, $\Phi_1(t, m)$ and $R_1(t, m)$ described above, which have been established theoretically (some only for $t > 0$), have been corroborated for different values

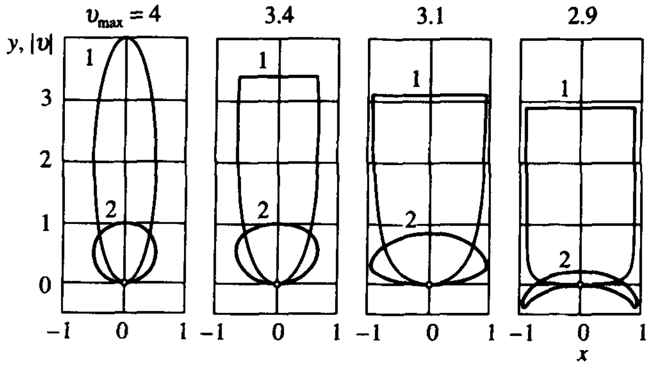

Fig. 2

of t by numerical experiments using the above analytical representations. Numerical experiments have shown that the system of equations (5.1) has no solutions for which $m > 1$, and each equation of system (5.1), considered separately for fixed t as an equation in m, has at most two roots – at most one root in each of the intervals specified in relations (5.3), with the exception of the interval $(1, \infty)$.

The system of equations (5.1) has been solved numerically, and computational experiments have confirmed that it is uniquely solvable. After finding all the parameters, it is not difficult to reproduce the shapes of the optimized airfoils.

Now let $\mu_2 = 0$ (recall that μ_0 and μ_2 cannot vanish simultaneously, and so $\mu_0 > 0$). By (4.2), we have

$$\upsilon^*(\gamma) = \{(\mu_0)^{-1}M(\gamma, \beta), \gamma \in [-\pi/2, t]; \ \upsilon_{max}, \gamma \in [t, \pi/2]\} \tag{5.4}$$

Thus, $\mu_0 = \upsilon_{max}^{-1}M(t, \beta)$ and $M(\gamma, \beta) \geq M(t, \beta)$ for all $\gamma \in [t, \pi/2]$. Since all the parameters except t are defined and their values must be such that Eqs (4.3) hold, we obtain two relations linking t, β and υ_{max}. Substituting expression (5.4) into Eqs (4.3), we obtain

$$\int_t^{\pi/2} \ln\frac{M(\gamma, \beta)}{M(t, \beta)}d\gamma = -\pi\ln\frac{M(t, \beta)}{\upsilon_{max}}, \quad \int_t^{\pi/2} \ln\frac{M(\gamma, \beta)}{M(t, \beta)}\sin\gamma d\gamma = 0 \tag{5.5}$$

As shown previously, if $\gamma \in [t, \pi/2]$ the first integrand in (5.5) is non-negative. Thus, the left-hand side of the first equality is always positive, while the left-hand side of the second is positive for $t \geq 0$ except when $t = \pi/2$. In that case the second relation of (5.5) becomes an identity, while the truth of the first relation in (5.5) can be guaranteed only at $\upsilon_{max} = 2(1 + \sin\beta)$. We have again obtained an optimum contour which is a circle (see Theorem 1).

Figure 2 shows the shapes of optimal airfoils and chord diagrams of velocity corresponding to the exact solution for $\varepsilon = 1$ at $\beta = 90°$ and different values of υ_{max}. The small circles on the contours represent coinciding branch points and rear stagnation points of the flow. It is obvious that the exact solutions have only a vertical axis of symmetry. Their characteristics are shown in table.

In Fig. 3 we show the exact shapes of the optimum airfoils (contours 2) and the corresponding chord diagram of velocity (curves 1) for different β values and fixed $\upsilon_{max} = 1.8$. The small circles on the contours represent critical points. Table 1 lists the characteristics of these exact solutions.

As noted in Theorem 1, for fixed $\beta = \beta^*$, the velocity υ_{max} must satisfy the necessary condition (2.5) for solvability. It would be interesting to investigate the tendency of the shape of the optimum airfoils to change as υ_{max} is increased, beginning from the value υ_{max}^* (the procedure described corresponds to motion along the vertical in the domain of admissible parameter values of the problem). Figure 4 implements these ideas for $\beta^* = 8°$ (see also the table). It is obvious that as υ_{max} increases the airfoils (contours 2) become thicker, the coefficient C_y increases, and the airfoils approach a disk, which they become at $\upsilon_{max} = \upsilon_{max}^{**}$, (in this example, υ_{max}^{**}, = 2.28). Conversely, as υ_{max} decreases, the airfoils become thinner and, beginning from a well-defined value of υ_{max}, become multiply-sheeted. Note that the leftmost airfoil in Fig. 4 is in effect a limiting case (when the value of υ_{max} is reduced further one obtains multiply-sheeted flow domains), though the cited values of maximum velocity are still far from the minimum possible value υ_{max}^* (in this case, $\upsilon_{max}^* = 1.15$). The chord velocity diagrams are labelled 1.

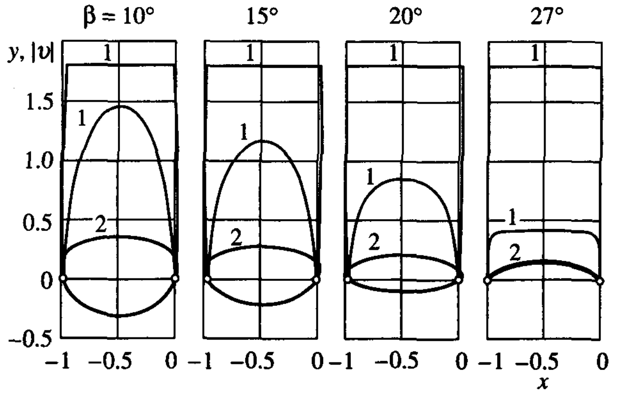

Fig. 3

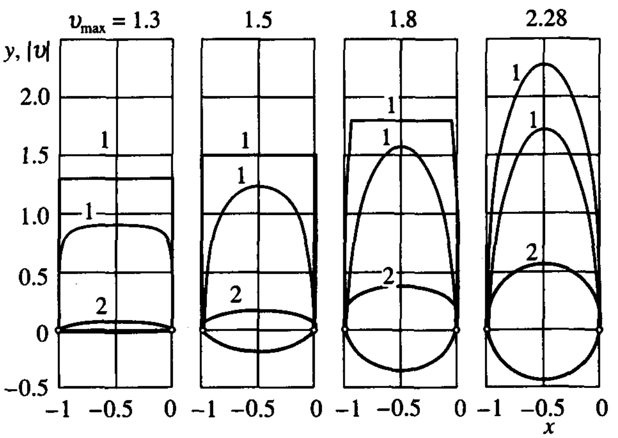

Fig. 4

Table 1

$\beta = 90°$			$\beta = 8°$			$\upsilon_{max} = 1.8$		
υ_{max}	t_{max}	C_y	υ_{max}	t_{max}	C_y	β	t_{max}	C_y
4	1	8	2.28	1	1.11	10°	0.66	1.37
3.4	1	7.95	1.8	0.73	1.1	15°	0.49	2
3.1	0.84	7.69	1.5	0.36	1.05	20°	0.3	2.53
2.9	0.22	6.62	1.3	0.09	0.94	27°	0.01	2.86

An analogous picture is observed if the velocity υ_{max} is given some fixed value in the admissible interval $[\upsilon^{*}_{max}, \upsilon^{**}_{max}]$ and β is increased up to its maximum admissible value β_{max} (this procedure corresponds to motion along the horizontal in the domain of admissible parameter values of the problem). Figure 3 illustrates this procedure for $\upsilon_{max} = 1.8$. It can be seen that as β increases the airfoils thicken, the coefficient C_y increases, the airfoils approach an arc of a circle and, beginning at some well-defined value of β much less than β_{max}, one obtains multiply-sheeted flow domains (in the example given, $\beta_{max} = 36°$).

Figure 5 illustrates optimum airfoil shapes (contour *2*) for $\varepsilon = 1$ and $\upsilon_{max} = 1.8$, $\beta = 28°$, corresponding to a multiply-sheeted flow domain, and also, in enlarged form, the structure of the contour in the neighbourhood of the rear stagnation point of the flow (by symmetry considerations, the structure of the optimized airfoil in the neighbourhood of the forward stagnation point is the same). Note that the velocity chord diagram (curve I) differs slightly from two “shelves”.

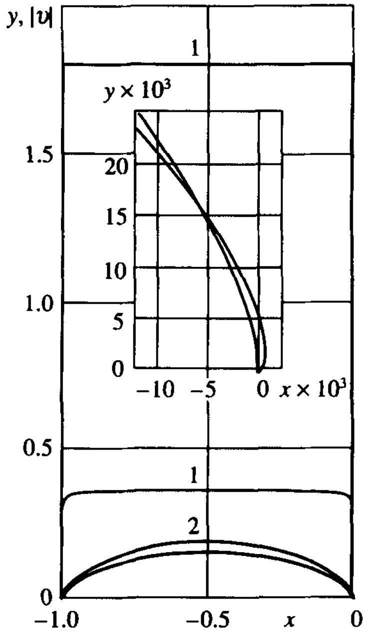

Fig. 5

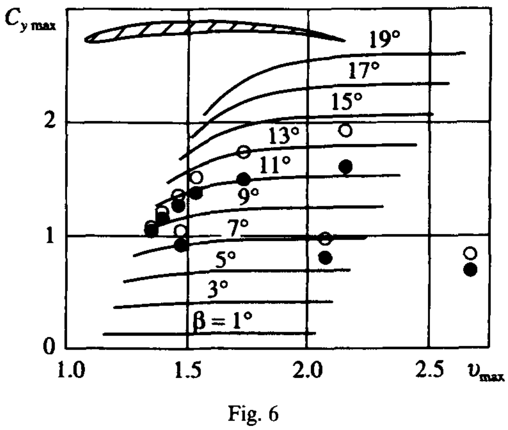

Fig. 6

Figure 6 represents the maximum value $C_{y\,max}$ of the lift coefficient as a function of υ_{max} for different β values, constructed on the basis of the exact solutions. The right end of each curve for a given β value corresponds to an exact solution of the problem in the form of a circle. As υ_{max} decreases, $C_{y\,max}$ becomes smaller, and for each β, in accordance with the necessary condition for the problem to be solvable (see Theorem 1), there is a minimum value $\upsilon^*_{max} = \exp\sin\beta$ of the maximum velocity on the contour. This value is represented by the left end of each curve. Thus, as υ_{max} decreases in its admissible interval of variation, the coefficients $C_{y\,max}$ decrease by at most 8%, and for each β there is a certain minimum value of the maximum velocity on the contour, which necessarily reaches the given value of υ_{max}.

We will now present the results of a comparison of the optimum airfoils obtained with a few known airfoils. Thus, the black dots in Fig. 6 represent values of the coefficient C_y computed for the Eppler airfoil E-61 (see, e.g. [16]); this airfoil has thickness of 6% and is also illustrated in Fig. 6) for a few values of υ_{max} and for those β for which the corresponding maximum coefficients $C_{y\,max}$ for the same υ_{max} values are represented by small circles in the graphs. The pairs of circles and black dots to the right of the graphs correspond to a comparison of the E-61 airfoil with a disk. As well see, the characteristics of the E-61 airfoil are fairly close to optimum.

6. ALLOWANCE FOR FLOW COMPRESSIBILITY AT SUBSONIC SPEEDS

As is well known, one approximate method to allow for compressibility is based on using the linear equations of gas dynamics, written in the hodograph plane of the velocity plane – Chaplygin's equations, which in the case of subsonic flow are

$$\frac{\partial\varphi}{\partial\theta} = K(S)^{1/2}\frac{\partial\psi}{\partial S}, \quad \frac{\partial\varphi}{\partial S} = -K(S)^{1/2}\frac{\partial\psi}{\partial\theta} \tag{6.1}$$

where φ is the velocity potential, ψ the stream function, θ is the velocity argument, and $K(\lambda)$ and $S(\lambda)$ are known functions of the reduced velocity $\lambda = \upsilon/a_*$, with a_* the critical velocity. In adiabatic motion of a perfect gas for which $p = \rho^\kappa$ (p is the pressure in units of the pressure at the forward stagnation point and κ is the adiabatic index)

$$\begin{gathered} K(\lambda) = (1-\lambda^2)\left(1-\frac{\lambda^2}{h^2}\right)^{-h^2}, \quad \rho(\lambda) = \left(1-\frac{\lambda^2}{h^2}\right)^{1/(\kappa-1)} \\ S(\lambda) = \ln\frac{2\lambda h}{(h^2-\lambda^2)^{1/2}+h(1-\lambda^2)^{1/2}} + h\ln\frac{(h^2-\lambda^2)^{1/2}+(1-\lambda^2)^{1/2}}{1+h}, \quad h^2 = \frac{\kappa+1}{\kappa-1} \end{gathered} \tag{6.2}$$

It is well known that for air ($\kappa = 1.41$) the function $K(\lambda)$ differs from unity at $\lambda \le 0.5$ by at most 1.6%; for such λ values, therefore, one can approximately put $K = 1$. As a result, the system of equations (6.1) yields the Cauchy–Riemann conditions, that is, the complex flow potential $w = \varphi + i\psi$ will be an analytic function of the variable $\chi = S - i\theta$. Under these conditions it follows from relations (6.2) that

$$\lambda(S) = \frac{\exp(S)}{1-c^2\exp(2S)}, \quad \rho(S) = \frac{1-c^2\exp(2S)}{1+c^2\exp(2S)} \tag{6.3}$$

where c^2 is a constant of integration, chosen to satisfy the condition of best approximation of the adiabatic functions by relations (6.3). Following the well-known approach of [17], one can take $c^2 = 0.296$ or $c^2 = [2(\kappa+1)(1-\lambda_\infty)]^{-1}$, where $\lambda_\infty = \upsilon_\infty/a_*$. The flow model thus obtained is known as the Chaplygin gas model. It guarantees satisfactory accuracy in computations of the velocity field λ in the subsonic domain. However [17], passage to a Chaplygin gas yields an error in the computation of the Mach numbers M from λ, since the Mach number for a Chaplygin gas does not formally reach unity; hence, when using the Chaplygin approximation, one usually determines only the velocity λ and then calculates M by the exact formula.

Within the limits of the Chaplygin gas model, the class of optimized contours is defined with the aid of a quasi-conformal mapping

$$z(\zeta) = \int_{e^{-i\beta}}^{\zeta} \exp[-\chi(\zeta)]w'(\zeta)d\zeta - c^2\overline{\exp[\chi(\zeta)]w'(\zeta)d\zeta} \tag{6.4}$$

with the substitution $\zeta = \exp(i\gamma)$. Here the control function is $P(\gamma) = \mathrm{Re}\chi(\exp(i\gamma))$ and

$$\frac{dw}{d\zeta} = u\left(1-\frac{\exp(-i\beta)}{\zeta}\right)\left(1+\frac{\exp(i\beta)}{\zeta}\right)$$

where the bar denotes complex conjugation. Note that when $c^2 = 0$ Eq. (6.4) yields a representation for the model of an ideal incompressible fluid. The expression (1.2) for the lift coefficient and the form of the solvability conditions remain the same, only the constants B_0, and B_1 and B_2 in Eq. (1.4) being replaced by

$$B_1 + iB_2 = \pi(1-\varepsilon) - 4\pi i\sin\beta\frac{c^2\Lambda_\infty^2}{1+c^2\Lambda_\infty^2}, \quad B_0 = 2\pi\ln\Lambda_\infty \tag{6.5}$$

where

$$\Lambda_\infty = \Lambda(\lambda_\infty), \quad \Lambda(\lambda) = 2\lambda[1 + (1 + 4c^2\lambda^2)^{1/2}]^{-1}$$

Thus, the class L of contours under consideration is given by integral representation (6.4), in which the control function $P(\gamma)$ satisfies constraints (1.4) (with the constants from (6.5) on the right), as well as constraint (1.3) with υ_{max} replaced by $2/(\sqrt{1+4c^2}+1)$. The analogue of the functional (1.3) is the following strictly convex functional (for a detailed derivation see [8])

$$J_c(P) = \int_0^{2\pi} \exp[-P(\tau)]\left[1 - c^2\exp[2P(\tau)]M^2(\tau, \beta)\left|2\sin\frac{\tau+\beta}{2}\right|^{2-2\varepsilon}\right]\left|2\sin\frac{\tau+\beta}{2}\right|^{\varepsilon-1}d\tau$$

which is identical with the functional (1.3) at $c = 0$ (corresponding to the transition to the model of an ideal incompressible fluid). Thus, in this case, maximization of the lift coefficient at fixed β requires minimizing the functional $J_c(P)$ over the set of admissible functions $P(\gamma)$ satisfying two linear constraints given by equalities, and one linear constraint which is an inequality. Unlike the case of ideal incompressible flow, in this situation the parameter β cannot take arbitrary values in the interval $[0, \pi/2]$; in fact, the following theorem holds.

Theorem 3 (see [13]). If

$$\lambda_\infty > \lambda_\infty^*(\beta), \quad \lambda_\infty^*(\beta) = \frac{\exp(r_*)}{1 - c^2\exp(2r_*)}$$

where r_* is the unique root of the equation

$$r - c_0 + D(r, \beta) = 0$$

$$c_0 = \ln\frac{2}{1+\sqrt{1+4c^2}}, \quad D(r, \beta) = \frac{\sin\beta[1 - c^2\exp(2r)]}{[1 + c^2\exp(2r)]}$$

then for any airfoil in a flow of Chaplygin gas having a theoretical angle of attack equal to or greater than β, there are points on the airfoil contour at which $\lambda > 1$.

The graph of the function $\lambda_\infty = \lambda_\infty^*(\beta)$ divides the domain of variation of the parameters λ_∞ and β into two zones. If it is known that the airfoil being designed is in a Chaplygin gas flow with velocity λ_∞ at infinity and theoretical angle of attack β such that the point (β, λ_∞) is above the aforementioned curve, then the airfoil produced by the Chaplygin gas model will have a supersonic zone. If the point (β, λ_∞) is below that curve, a subcritical airfoil with the specified characteristics can be designed.

We will now construct an exact solution of the problem in the case when $\varepsilon = 1$, following the scheme outlined above. The extended functional has the same form, except that $J_0(P)$ is replaced by $J_c(P)$, and the form of the extremal function $P^*(\gamma)$ is determined from the necessary condition for the functional to have an extremum:

$$P^*(\gamma) = -\ln\frac{g(\mu_k, \mu; \gamma) + \sqrt{g^2(\mu_k, \mu; \gamma) - 4c^2M^2(\gamma, \beta)}}{2} \tag{6.6}$$

The function $\mu^*(\gamma)$ corresponding to the exact solution is now

$$\mu^*(\gamma) = \max\left\{0, \sqrt{1+4c^2}M(\gamma, \beta) - \mu_0 - \mu_1\cos\gamma - \mu_2\sin\gamma\right\} \tag{6.7}$$

and the velocity distribution corresponding to the extremal function $P^*(\gamma)$ is

$$|\lambda^*(\gamma)| = \min\left\{1, \left|\frac{M(\gamma, \beta)}{\sqrt{g^2(\mu_0, 0; \gamma) - 4c^2M^2(\gamma, \beta)}}\right|\right\}$$

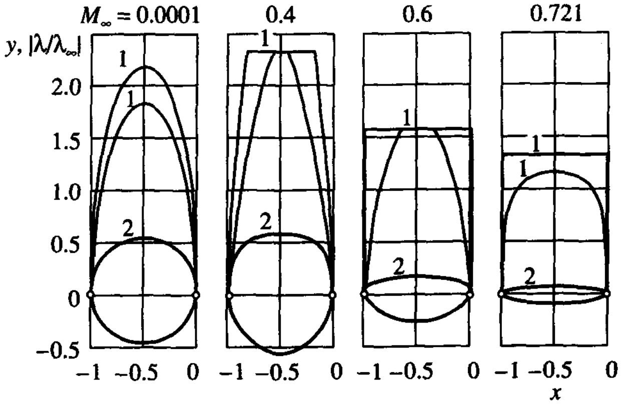

Fig. 7

These expressions naturally yield formulae (3.2) and (3.3) if $c^2 = 0$.

The system of equations in μ_0, μ_1 and μ_2 is now derived from relations (1.4) by substituting the extremal function, with due note of expression (6.7), with the appropriate values of B_0, B_1 and B_2

$$\int_0^{2\pi} R(\mu_k, \mu^*(\gamma); \gamma) d\gamma = B_0$$

$$\int_0^{2\pi} R(\mu_k, \mu^*(\gamma); \gamma) e^{i\gamma} d\gamma = B_1 + iB_2$$

$$R(\mu_k, \mu^*(\gamma); \gamma) \equiv \ln \frac{g(\mu_k, \mu^*(\gamma); \gamma) + \sqrt{g^2(\mu_k, \mu^*(\gamma); \gamma) - 4c^2 M^2(\gamma, \beta)}}{2}$$

Note that the function (6.7), as in the case of an ideal incompressible fluid, has the symmetry property

$$\mu^*(\pi - \gamma)_{\mu_1 = \alpha} = \mu^*(\gamma)_{\mu_1 = -\alpha}$$

When $\varepsilon = 1$ this property, combined with the fact that $B_1 = 0$, enables one to prove, as in the ideal incompressible fluid model, that the optimum solution is symmetric.

We have thus derived exact solutions of the fundamental variational inverse boundary-value problem of aerodynamics for the Chaplygin gas model, as well as a system of equations to determine the parameters. Computations and results obtained in the case under consideration by constructing exact solutions for different choices of β and $\upsilon_{\max}$ values have confirmed the tendencies of the optimized shapes to vary, as shown and described above in the context of the ideal incompressible fluid model.

Figure 7 illustrates exact solutions (contours *2*) corresponding to $\varepsilon = 1$ and $\beta = 5°$ at different Mach numbers M_∞, as well as chord diagrams of the velocity (curves *1*). It is interesting to note that in some cases each of the extremal velocity distributions has two "shelves" (the cases $M_\infty = 0.4$ and $M_\infty = 0.6$ in Fig. 7). This effect has not been observed for the ideal incompressible fluid model.

We wish to thank A. N. Ikhsanova for carrying out the computations and for the results of numerical experiments report in the paper.

This research was supported financially by the Russian Foundation for Basic Research (03-01-00015) and the "Universities of Russia" programme (UR 04.01.009).

REFERENCES

1. HASLINGER, J. and NEITTAANMÄKI, P., *Finite Element Approximation for Optimal Shape Design: Theory and Application*. Wiley, Chichester, 1988.

2. SIRAZETDINOV, T. K., *Optimization of Systems with Distribution Parameters*. Nauka, Moscow, 1977.
3. LAVRENT'YEV, M. A., *An extremal problem in airfoil theory*. Trudy TsAGI im. N. Ye. Zhukovskogo, Issue 155, 1934.
4. ZUBOV, V. I., The problem of the optimum airfoil in an ideal incompressible flow. *Zh. Vyschisl. Mat. Mat. Fiz.*, 1980, **20**, 1, 241–245.
5. YELIZAROV, A. M., Some extremal problems of airfoil theory. *Izv. Vuzov. Matematika*, 1988, 10, 71–74.
6. YELIZAROV, A. M. and FEDOROV, Ye. V., Optimization of aerodynamic shapes by the method of inverse boundary-value problems. *Prikl. Mat. Mekh.*, **54**, 4, 571–580. 1990.
7. YELIZAROV, A. M. and FEDOROV, Ye. V., Solution of variational inverse boundary-value problems of aerodynamics by numerical optimization methods. *Zh. Prikl. Mekh. Tekhn. Fiz.*, 2, 73–81.
8. YELIZAROV, A. M., FEDOROV, Ye. V. and FOKIN, D. A., Variational inverse boundary-value problems of aerodynamics for subsonics gas flow. *Zh. Vyschisl. Mat. Mat. Fiz.*, 1993, **33**, 6, 958–968.
9. YELIZAROV, A. M., IL'INSKII, N. B. and POTASHEV, A. V., *Inverse Boundary-Value Problems of Aerodynamics*. Nauka, Moscow, 1994.
10. FOKIN, D. A., Maximization of the aerodynamic quality of airfoils with a turbulent boundary layer. *Izv. Ross. Akad. Nauk. MZhG*, 1998, 3, 177–184.
11. ELIZAROV, A. M. and FOKIN, D. A., Upper estimates of airfoil aerodynamic characteristics for a viscous incompressible flow. *ZAMM*, 1999, **79**, 11, 757–762.
12. YELIZAROV, A. M. and FOKIN, D. A., Variational inverse boundary-value problems of aerodynamics. *Dokl. Ross. Akad. Nauk*, 2001, **377**, 6, 758–763.
13. YELIZAROV, A. M., IL'INSKII, N. B., POTASHEV, A. V. and STEPANOV, G. Yu., *Basic Methods, Results, Applications and Unsolved Problems of the Theory of Inverse Boundary-Value Problems of Aerodynamics*. Trudy Mat. Tsentra im. N. I. Lobachevskogo, Vol. 10. Izd. DAS, Kazan, 2002.
14. LAVRENT'YEV, M. A. and SHABAT, B. V., *Methods of the Theory of Functions of a Complex Variable*. Nauka, Moscow, 1987.
15. IOFFE, F. D. and TIKHOMIROV, V. M., *Theory of Extremal Problems*. Nauka, Moscow, 1974.
16. EPPLIER, R., *Airfoil Design and Data*. Springer, Berlin, 1990.
17. STEPANOV, G. Yu., *Hydrodynamics of Turbomachine Cascades*. Fizmatgiz, Moscow, 1962.

Translated by D.L.